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HARMONIC WAVES IN THREE-DIMENSIONAL ELASTIC COMPOSITES†

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Abstract—For a periodic elastic composite which consists of a matrix and fibers with finite dimensions (i.e. a three-dimensional problem), here are given estimates for eigenfrequencies and eigenfunctions. Calculations are based on a new quotient which has been proposed by Nemat-Nasser. The periodic character of the eigenfrequencies is pointed out, and illustrative examples are given.

1. INTRODUCTION

A mixed variational method (i.e. a modified Reissner's variational principle) has been proposed by one of the authors for the analysis of harmonic waves propagating in one-, two- and three-dimensional elastic composites which consist of identical unit cells that are repeatedly extended in all directions [1–3]. The corresponding eigenfrequencies and their lower and upper bounds have been calculated in [4, 5] for a number of illustrative cases.

Since the equations that appear in these analyses have a form analogous to that of the Schrödinger equation, with coefficients having periodicity similar to that which occurs in the case of lattice dynamics, it is natural to examine the composite problem by the approach used in the analysis of the lattice dynamics; for discussions pertaining to lattice dynamics, see, for example, [6, 7].§

In this paper the problem of harmonic waves in a *three-dimensional* elastic composite will be examined from this standpoint, and explicit numerical results for a three-dimensional composite will be given. The eigenfrequencies are periodic functions of the wave vector, where one period is contained in each "reciprocal cell." As will be seen, it proves more convenient to employ the so-called "elementary zone," instead of the recirpocal cell, in order to depict the frequencies.

2. STATEMENT OF THE PROBLEM

We consider three-dimensional composites which consist of, say, hexahedron unit cells defined by means of three base vectors $\boldsymbol{\ell}^\beta$, $\beta = 1, 2, 3$, where each cell is composed of two elastic constituents, the matrix and the fibers. The position vector of a point in the cell is given by

$$\mathbf{r} = \xi_\beta \boldsymbol{\ell}^\beta, \qquad \beta = 1, 2, 3, \tag{2.1}$$

where $0 \le \xi_\beta \le 1$, and the summation convention on repeated indices is used. The "reciprocal" base vectors are then defined by

$$\mathbf{d}^1 = \frac{2\pi}{\Delta}\boldsymbol{\ell}^2 \times \boldsymbol{\ell}^3, \quad \mathbf{d}^2 = \frac{2\pi}{\Delta}\boldsymbol{\ell}^3 \times \boldsymbol{\ell}^1, \quad \mathbf{d}^3 = \frac{2\pi}{\Delta}\boldsymbol{\ell}^1 \times \boldsymbol{\ell}^2, \tag{2.2}$$

where $\Delta = \boldsymbol{\ell}^1 \cdot \boldsymbol{\ell}^2 \times \boldsymbol{\ell}^3$. Three vectors $\mathbf{d}^1$, $\mathbf{d}^2$ and $\mathbf{d}^3$ also define a cell which we call the "reciprocal cell." Figure 1 shows a cell and its reciprocal cell.

For harmonic waves of frequency ω, the equations of motion are

$$\sigma_{jk,k} + \lambda\rho u_j = 0, \quad \sigma_{jk} = C_{jkmn}u_{m,n}, \quad j, k, m, n = 1, 2, 3, \tag{2.3}$$

where $\lambda = \omega^2$; $\sigma_{jk}\, e^{\pm i\omega t}$ and $u_j\, e^{\pm i\omega t}$ are the stress and displacement fields; $i = \sqrt{-1}$; t is the time;

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§References to works on waves in elastic composites can be found, for example, in [8, 9].

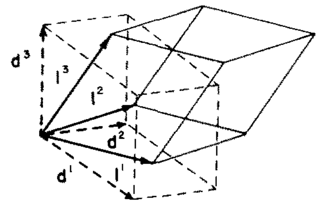

Fig. 1. A unit cell and its reciprocal.

and a comma followed by an index denotes partial derivative with respect to the corresponding coordinate. The mass-density ρ and the elasticity tensor C_{jkmn} are continuous and, say, continuously differentiable functions of $\mathbf{x}$ in the subregions occupied by the matrix and by the fibers, but, in general, admit finite discontinuities at the boundaries between these two constituents. Let these interior boundaries be denoted by Σ. Then one must impose the following continuity condition for the tractions:

$$[\sigma_{jk}(\mathbf{x}^+) - \sigma_{jk}(\mathbf{x}^-)]n_k = 0, \quad \mathbf{x} \text{ on } \Sigma, \tag{2.4}$$

where n_k is the unit normal on Σ, pointing from one subregion, say, subregion 1, to the adjacent one, say, subregion 2, $\sigma_{jk}(\mathbf{x}^+)$ is the limiting value of the stress as a point on Σ is approached along the normal from within subregion 1, and $\sigma_{jk}(\mathbf{x}^-)$ is the limit when this point is appraoched from within subregion 2.

Let $\mathbf{q}$ denote the wave vector. Then the boundary conditions for harmonic waves become

$$\begin{aligned} u_j(\mathbf{x} + \ell^\beta) &= u_j(\mathbf{x})\, e^{i\mathbf{q}\cdot\ell^\beta} \\ t_j(\mathbf{x} + \ell^\beta) &= -t_j(\mathbf{x})\, e^{i\mathbf{q}\cdot\ell^\beta}, \quad \text{for } \mathbf{x} \text{ on } \partial\mathcal{R}, \end{aligned} \tag{2.5}$$

where $\partial\mathcal{R}$ denotes the boundary of the unit cell, and $t_j = \sigma_{jk}n_k$ is the traction vector, where n_k is the exterior unit normal on $\partial\mathcal{R}$.

It can be verified that the solutions of eqns (2.3) that satisfy (2.5) make the functional

$$\lambda_N = \frac{\langle \sigma_{jk}, u_{j,k}\rangle + \langle u_{j,k}, \sigma_{jk}\rangle - \langle D_{jkmn}\sigma_{jk}, \sigma_{mn}\rangle}{\langle \rho u_j, u_j\rangle} \tag{2.6}$$

stationary, where D_{jkmn} is the elastic compliance tensor, and

$$\langle gu, v\rangle \equiv \int_{\mathcal{R}} guv^*\, dV,$$

g being a real-valued weighting function, star denoting the complex conjugate, and the integration extends over the entire cell. Functional (2.6) has been called the *new quotient*, see [1–5].

3. PERIODIC STRUCTURE OF SOLUTION

For approximation, consider the test functions

$$\bar{u}_j = \sum_{\alpha,\beta,\gamma=0}^{M} U_j^{\alpha\beta\gamma} f^{\alpha\beta\gamma}, \quad \bar{\sigma}_{jk} = \sum_{\alpha,\beta,\gamma=0}^{M} S_{jk}^{\alpha\beta\gamma} f^{\alpha\beta\gamma}, \tag{3.1}$$

where $f^{\alpha\beta\gamma}$ are the coordinate functions.† Substitute from (3.1) into (2.6), and set equal to zero the derivatives of λ_N with respect to the unknown coefficients $U_j^{\alpha\beta\gamma}$ and $S_{jk}^{\alpha\beta\gamma}$ in order to obtain a set of linear equations from which estimates for the eigenvalues λ_N and eigenfunctions $\bar{u}_j$ and $\bar{\sigma}_{jk}$ can be obtained.

When the coordinate functions are of the form

†These functions satisfy (2.5).

$$f^{\alpha\beta\gamma}(\mathbf{q},\mathbf{r}) = \exp\left[i\{\mathbf{q}\cdot\mathbf{r} + 2\pi(\alpha\xi_1 + \beta\xi_2 + \gamma\xi_3)\}\right], \qquad \alpha, \beta, \gamma = 0, \pm 1, \pm 2, \ldots, \pm M', \tag{3.2}$$

one observes by setting

$$\mathbf{q}' = \mathbf{q} + h_1\mathbf{d}^1 + h_2\mathbf{d}^2 + h_3\mathbf{d}^3, \tag{3.3}$$

(for integer-valued h_1, h_2, and h_3) and substituting into (3.2), that

$$f^{\alpha\beta\gamma}(\mathbf{q}',\mathbf{r}) = \exp\left[i\{\mathbf{q}\cdot\mathbf{r} + 2\pi[(\alpha + h_1)\xi_1 + (\beta + h_2)\xi_2 + (\gamma + h_3)\xi_3]\}\right] \tag{3.4}$$

which shows that the coordinate functions corresponding to the wave vector $\mathbf{q}'$ are the same as those for $\mathbf{q}$, except for the indices α, β, and γ which are changed to $\alpha + h_1$, $\beta + h_2$, and $\gamma + h_3$, respectively.

When M' is sufficiently large and h_1, h_2, and h_3 are suitably small integers, the sets of coordinate functions (3.2) and (3.4) are essentially equivalent, i.e. they lead to essentially the same estimates for the lower eigenvalues. At the limit when $M' \to \infty$, the two sets of coordinate functions become equivalent and therefore the estimated eigenvalues coincide. In this case, for $\mathbf{q}$ and $\mathbf{q}'$, the eigenvalues are equal. This means that the eigenvalues are periodic functions of $\mathbf{q}$, with one period being contained in one reciprocal cell.† In approximate calculations, however, M' cannot be too large, and therefore the estimated eigenvalues are not periodic in regions far from the origin in the $\mathbf{q}$-space; in any case, h_1, h_2, and h_3 must not exceed M'.

Since the eigenfrequencies are periodic functions of the wave vector, and since one reciprocal cell contains one period of all frequencies, one needs only to estimate the frequencies as functions of the wave vector in one reciprocal cell. However, in order to include all possible wave vectors (i.e. $\mathbf{q}$ and $-\mathbf{q}$, for example), one considers the so-called *elementary zone*. For the two-dimensional case, this is shown in Fig. 2. Here the elementary zone is obtained by drawing perpendicular bisectors of lines joining the origin to each of the other vertices of the reciprocal cell. All information for harmonic waves propagating through the fiber-reinforced composite with the given reciprocal cell can now be stated in terms of the eigenfrequencies on this polygonal zone, where an eigenfrequency ν as a function of $\mathbf{q}$, is depicted as a surface over the elementary zone, i.e. one surface for each eigenfrequency. This is similar to the Fermi surface in, say, metals and superconductors.

For three-dimensional case, the elementary zone is a volume with the origin of the reciprocal cell at its center. The boundary of the zone is defined by planes which bisect the vectors drawn from the origin to each vertex of the reciprocal cell. A set of such surfaces is given for each intersection of the elementary zone by a plane that passes through the origin. If one intersects these surfaces by a plane perpendicular to this intersection and passing through the origin, then one obtains branches similar to those shown in Fig. 3(a).

The number of branches that are estimated depends on the value of M', i.e. on the number of coordinate functions used in the approximate analysis; for the exact solutions, there are infinitely many branches. For a given M', there are $3(2M'+1)^3$ branches for a three-dimensional composite. The three lowest branches are called the acoustical branches, while the higher ones

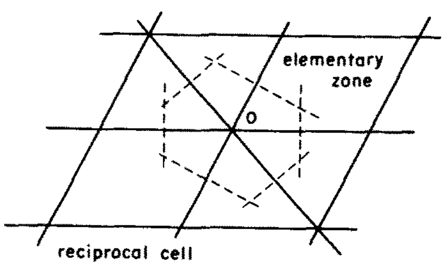

Fig. 2. A reciprocal cell and the corresponding elementary zone.

†Note that the same coordinate functions can be used in the usual Rayleigh quotient which is expressed in terms of u_i, and hence the present argument does not depend on whether the new quotient or the usual Rayleigh quotient is being used; see [5].

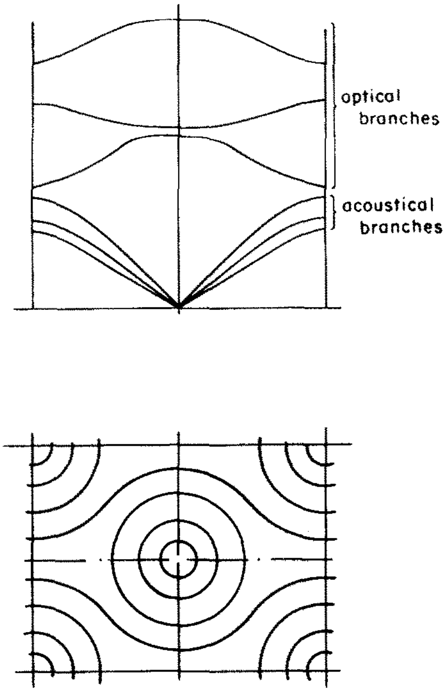

Fig. 3. (a) Acoustical and optical branches. (b) Loci of constant ν on an elementary zone.

are called the optical branches. In the two-dimensional case, with M' fixed, one estimates only $2(2M'+1)^2$ branches, the lowest two being acoustical, and the others optical. Finally, when one deals with a one-dimensional case, there is only one acoustical branch.

It is common to represent the loci of points of constant ν on the elementary zone, as in Fig. 3b. For a homogeneous material, ν becomes a linear function of **q**, and the curves in Fig. 3a reduce to straight lines.

4. NUMERICAL EXAMPLES

For illustration consider a composite whose unit cell is a rectangular parallelepiped with vectors ℓ^{β} parallel to the coordinate axes. Let the cell consist of two material constituents, the matrix, and the fiber which is in the shape of a rectangular parallelepiped or an ellipsoid. The fiber is centered with respect to the cell in such a manner that, when it is a rectangular parallelepiped, then its faces are parallel to those of the cell, and when it is an ellipsoid, then its three axes are each placed parallel to the corresponding sides of the cell (see Fig. 4).

If one substitutes from (3.1) and (3.2) into (2.6), and then renders λ_N stationary with respect to the variation of the unknown coefficients in (3.1), one obtains

$$\left[H\Phi^{-1}H^{*}-\frac{\nu^2}{d}\Omega\right]U=0, \tag{4.1}$$

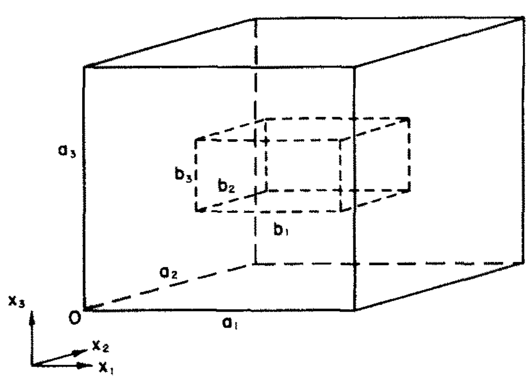

Fig. 4. A cell with a rectangular parallelepiped fiber.

where matrices H, Φ and Ω are defined as follows:

$$H = \begin{bmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & 0 & 0 & 0 \\ 0 & \mathcal{H}_1 & 0 & \mathcal{H}_2 & \mathcal{H}_3 & 0 \\ 0 & 0 & \mathcal{H}_1 & 0 & \mathcal{H}_2 & \mathcal{H}_3 \end{bmatrix}, \tag{4.2}$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are $(2M'+1)^3 \times (2M'+1)^3$ matrices defined in the following manner: For $\alpha = \delta$, $\beta = \mu$ and $\gamma = \tau$, and with $Q_j = q_j a_j$ (no sum),

$$\mathcal{H}_1(I_1, J_1) = -i(Q_1 + 2\pi\alpha), \quad \mathcal{H}_2(I_1, J_1) = -i(Q_2 + 2\pi\beta)n_0,$$
$$\mathcal{H}_3 = -i(Q_3 + 2\pi\gamma)m_0. \tag{4.3}$$

For $\alpha \neq \delta$, $\beta \neq \mu$, or $\gamma \neq \tau$,

$$\mathcal{H}_1(I_1, J_1) = \mathcal{H}_2(I_1, J_1) = \mathcal{H}_3(I_1, J_1) = 0, \tag{4.4}$$

where $I_1 = = (\alpha + 1 + M') + (\beta + M')(2M'+1) + (\gamma + M')(2M'+1)^2$, and $J_1 = (\delta + 1 + M') + (\mu + M')(2M'+1) + (\tau + M')(2M'+1)^2$, $\delta, \mu, \tau = 0, \pm 1, \pm 2, \ldots, \pm M'$.

$$\Omega = \begin{bmatrix} \underline{\Omega} & 0 & 0 \\ 0 & \underline{\Omega} & 0 \\ 0 & 0 & \underline{\Omega} \end{bmatrix}, \tag{4.5}$$

and

$$\Phi = \begin{bmatrix} \Delta_{1111} & 0 & 0 & \Delta_{1122} & 0 & \Delta_{1133} \\ 0 & 4\Delta_{1212} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\Delta_{1313} & 0 & 0 & 0 \\ \Delta_{1122} & 0 & 0 & \Delta_{2222} & 0 & \Delta_{2233} \\ 0 & 0 & 0 & 0 & 4\Delta_{2323} & 0 \\ \Delta_{1133} & 0 & 0 & \Delta_{2233} & 0 & \Delta_{3333} \end{bmatrix} \tag{4.6}$$

where $\underline{\Omega}$ and $\Delta_{ijk\ell}$ are $(2M'+1)^3 \times (2M'+1)^3$ matrices defined in the following manner:

(a) for composites with rectangular fibers,

$$\begin{aligned}
\underline{\Omega}(I_1, J_1) &= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\delta-\alpha)]}{\pi(\delta-\alpha)} \frac{\sin[\pi(\mu-\beta)]}{\pi(\mu-\beta)} \frac{\sin[\pi(\tau-\gamma)]}{\pi(\tau-\gamma)} && \text{if } \delta \neq \alpha, \mu \neq \beta \text{ and } \tau \neq \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\delta-\alpha)}{\pi(\delta-\alpha)} \frac{\sin[\pi(\mu-\beta)]}{\pi(\mu-\beta)} \ell_2 && \text{if } \delta \neq \alpha, \mu \neq \beta \text{ and } \tau = \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\mu-\beta)]}{\pi(\mu-\beta)} \frac{\sin[\pi(\tau-\gamma)]}{\pi(\tau-\gamma)} n_2 && \text{if } \delta = \alpha, \mu \neq \beta \text{ and } \tau \neq \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\delta-\alpha)]}{\pi(\delta-\alpha)} \frac{\sin[\pi(\mu-\beta)]}{\pi(\mu-\beta)} m_2 && \text{if } \delta \neq \alpha, \mu = \beta \text{ and } \tau \neq \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\delta-\alpha)]}{\pi(\delta-\alpha)} m_2\ell_2 && \text{if } \delta \neq \alpha, \mu = \beta \text{ and } \tau = \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\mu-\beta)]}{\pi(\mu-\beta)} n_2\ell_2 && \text{if } \delta = \alpha, \mu \neq \beta \text{ and } \tau = \gamma; \\
&= \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{\sin[\pi(\tau-\gamma)]}{\pi(\tau-\gamma)} n_2 m_2 && \text{if } \delta = \alpha, \mu = \beta \text{ and } \tau \neq \gamma; \\
&= 1 && \text{if } \delta = \alpha, \mu = \beta \text{ and } \tau = \gamma;
\end{aligned} \tag{4.7}$$

(b) for composites with ellipsoidal fibers,

$$\Omega(I_1, J_1) = 1 \quad \text{if } \alpha = \delta, \beta = \mu \text{ and } \gamma = \tau;$$
$$= \left(\frac{\pi}{2}\right)^{3/2} \frac{\theta - 1}{\bar{n}_1 + \bar{n}_2\theta} \frac{n_2 m_2 \ell_2 J_{3/2}(R)}{R^{3/2}} \quad \text{otherwise,} \tag{4.8}$$

where

$$R = \pi\{n_2^2(\delta - \alpha)^2 + m_2^2(\mu - \beta)^2 + \ell_2^2(\tau - \gamma)^2\}^{1/2}. \tag{4.9}$$

For $I_1 \neq J_1$, $\Delta_{ijk\ell}$ is obtained if one substitutes $(\gamma_{ijk\ell} - 1)R_{ijk\ell}/(\bar{n}_1 + \bar{n}_2\gamma_{1111})$ for $(\theta - 1)/(\bar{n}_1 + \bar{n}_2\theta)$ in the expression for $\Omega(I_1, J_1)$ and, for $I_1 = J_1$, one has

$$\Delta_{ijk\ell} = \frac{(\bar{n}_1 + \bar{n}_2\gamma_{ijk\ell})R_{ijk\ell}}{(\bar{n}_1 + \bar{n}_2\gamma_{1111})}.$$

In the above expressions, the following notation is used:†

$$\nu^2 = \frac{\omega^2 a_1^2 \bar{\rho}}{\bar{C}_{1111}}, \quad \bar{\rho} = \rho^{(1)}\bar{n}_1 + \rho^{(2)}\bar{n}_2, \quad \bar{C}_{1111} = C^{(1)}_{1111}\bar{n}_1 + C^{(2)}_{1111}\bar{n}_2,$$
$$\bar{n}_1 = 1 - \bar{n}_2, \quad \bar{n}_2 = \frac{b_1 b_2 b_3}{a_1 a_2 a_3}, \quad \theta = \frac{\rho^{(2)}}{\rho^{(1)}}, \quad n_2 = \frac{b_1}{a_1}, \quad m_2 = \frac{b_2}{a_2},$$
$$\ell_2 = \frac{b_3}{a_3}, \quad n_0 = \frac{a_1}{a_2}, \quad m_0 = \frac{a_1}{a_3}, \quad \gamma_{jk\ell m} = \frac{D^{(2)}_{jk\ell m}}{D^{(1)}_{jk\ell m}},$$
$$R_{jk\ell m} = \frac{D^{(1)}_{jk\ell m}}{D^{(1)}_{1111}}, \quad d = 1/(\bar{C}_{1111}\bar{D}_{1111}), \quad \bar{D}_{1111} = D^{(1)}_{1111}\bar{n}_1 + D^{(2)}_{1111}\bar{n}_2 \tag{4.10}$$

where $j, k, \ell, m = 1, 2, 3$ and b_1, b_2 and b_3 are the lengths of the sides of the fiber when it is a rectangular parallelepiped, and the diameters when it is an ellipsoid, which are placed parallel to the x_1-, x_2- and x_3-axes, respectively.

In Fig. 5(a) the eigenfrequencies are plotted for $q_2 = q_3 = 0$, as functions of the wave number q_1 for a three-dimensional composite reinforced by infinite fibers of the square-shaped cross section. The lowest three branches are the acoustical branches which correspond to the shear vertical (SV), shear horizontal (SH), and longitudinal (P) waves. In Fig. 5(b), a map of the lowest eigenfrequency (SV) is potted on an elementary zone, using lines of equal ν for the case of $q_3 = 0$. Figure 6 shows the eigenfrequencies vs wave number curves for a three-dimensional composite with rectangular fibers and Fig. 7 with ellipsoidal fibers.

5. THE STESS RAYLEIGH QUOTIENT

The stress Rayleigh quotient

$$\bar{\lambda}_R = \frac{\langle R\sigma_{jk,k}, \sigma_{j\ell,\ell}\rangle}{\langle D_{jkmn}\sigma_{jk}, \sigma_{mn}\rangle}, \qquad R = 1/\rho, \tag{5.1}$$

gives non-zero estimates for the eigenvalues, provided that the test functions σ_{jk} are (at least approximately) compatible in the sense that they correspond (approximately) to a suitable displacement field. In [5] it was *incorrectly* stated without proof that if one uses in (5.1) the test function

$$\sigma_{ij} = \sum_{p=1}^{\bar{M}} \xi_p \bar{\sigma}^{(p)}_{jk}, \tag{5.2}$$

where ξ_p's are unknown coefficients, and $\bar{\sigma}^{(p)}_{jk}$ are stress fields obtained by the aid of the new quotient (2.6) in the manner discussed in the preceding sections, then minimization with respect to ξ_p's yields upper bounds for the exact eigenfrequencies. It is not difficult to show, by

†For ellipsoidal fibers, $\bar{n}_2$ must be replaced by $\frac{\pi}{6}\frac{b_1 b_2 b_3}{a_1 a_2 a_3}$.

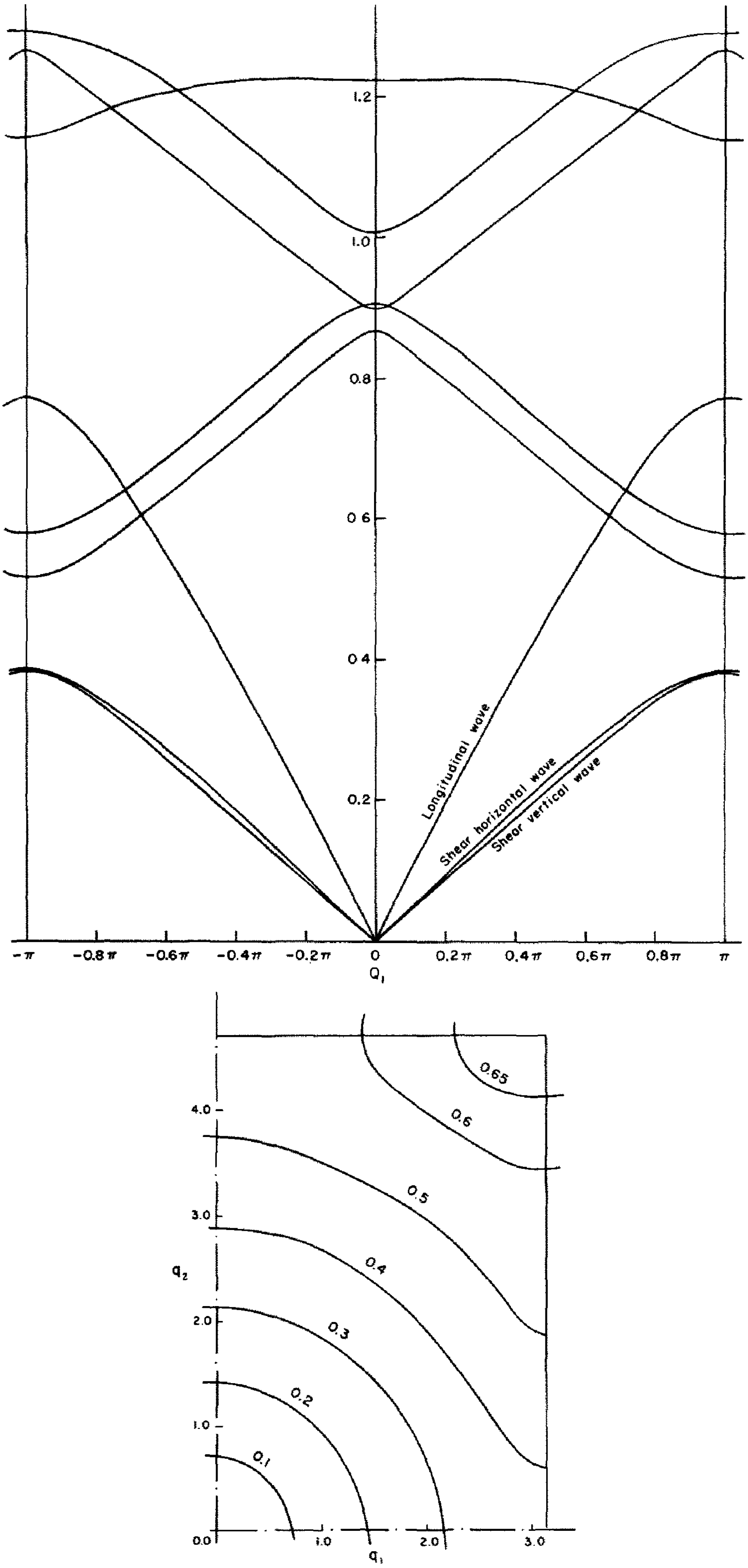

Fig. 5. (a) Acoustical and optical brnaches for waves propagating perpendicular to fibers (square fibers): $\theta = 3$, $\gamma = 100$, $\nu_M = 0.35$, $\nu_F = 0.3$, $M' = 1$, $n_0 = 2/3$, $m_0 = 1$, $n_2 = 0.5$, $m_2 = 1/3$, $\ell_2 = 1$. (b) Equal ν curves for the lowest eigenfrequency: $q_3 = 0$.

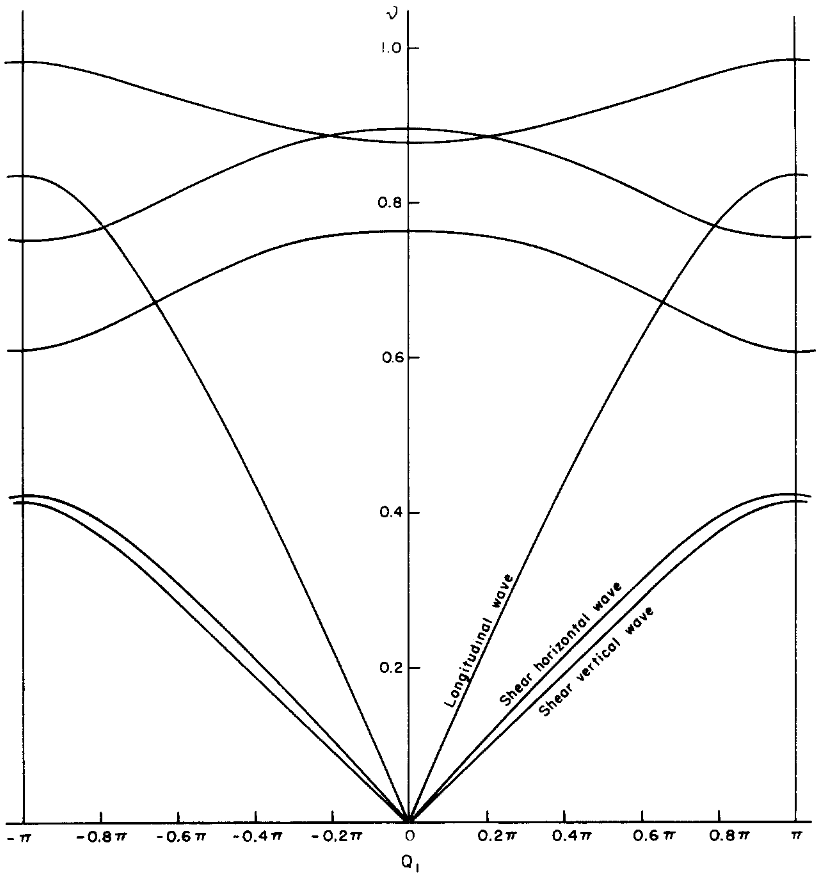

Fig. 6. Acoustical and optical branches for waves propagating in a composite with rectangular inclusions ($q_2 = q_3 = 0$): $\theta = 3$, $\gamma = 50$, $\nu_M = 0.35$, $\nu_F = 0.3$, $n_0 = 1$, $m_0 = 1$, $n_2 = 0.8$, $m_2 = 0.9$, $\ell_2 = 0.7$, $M' = 1$.

counterexample, that this claim is not, in general, valid. However, (5.1) can be quite useful if the mass-density is smooth or has a relatively small discontinuity, which is often the case in practice. In such a situation one can use the stress fields $\bar{\sigma}_{jk}^{(p)}$, obtained from the new quotient, as the coordinate functions in (5.2), and then minimize (5.1). Since $\bar{\sigma}_{jk}^{(p)}$ are *approximately* compatible field, this will not result in zero eigenvalues; if test functions $(3.1)_2$ are used *directly* in (5.1), and then $\bar{\lambda}_R$ is minimized, one obtains zero values for the first M^3 eigenvalues. Moreover, $\bar{M}$ in (5.2) can be chosen smaller than $M = 2M' + 1$ without substantially altering the accuracy of the results. For example, when $M' = 1$, one obtains from the new quotient $(3(2M' + 1)^3 = 81$ eigenfunctions $\bar{\sigma}_{jk}^{(p)}$. To improve the estimated, say, first 10 eigenvalues, one only needs to use $\bar{M} \geq 10$ in (5.2). These first 10 stress fields are approximately compatible, since they all satisfy the following condition:

$$\langle D_{jkmn}\bar{\sigma}_{jk}^{(p)} - \bar{u}_{m,n}^{(p)},\ f^{\alpha\beta\gamma}\rangle = 0, \quad \alpha, \beta, \gamma = 1, 2, \ldots, M, \tag{5.3}$$

Table 1. The lowest four eigenfrequencies by the method of the new quotient, and the improved eigenfrequencies for a wave guide: $\theta = 3$, $\gamma = 50$, $\nu_M = 0.35$, $Q = 1$; $\nu_F = 0.3$, $n0 = m_0 = 1.0$, $\bar{n}_2 = 0.8$; $M' = 1$, $\bar{M} = 9$

Eigenvalues by new quotient	Improved eigenvalues	Exact solution
0.205	0.214	0.248
0.472	0.494	—
0.810	0.852	0.912
1.833	2.021	—

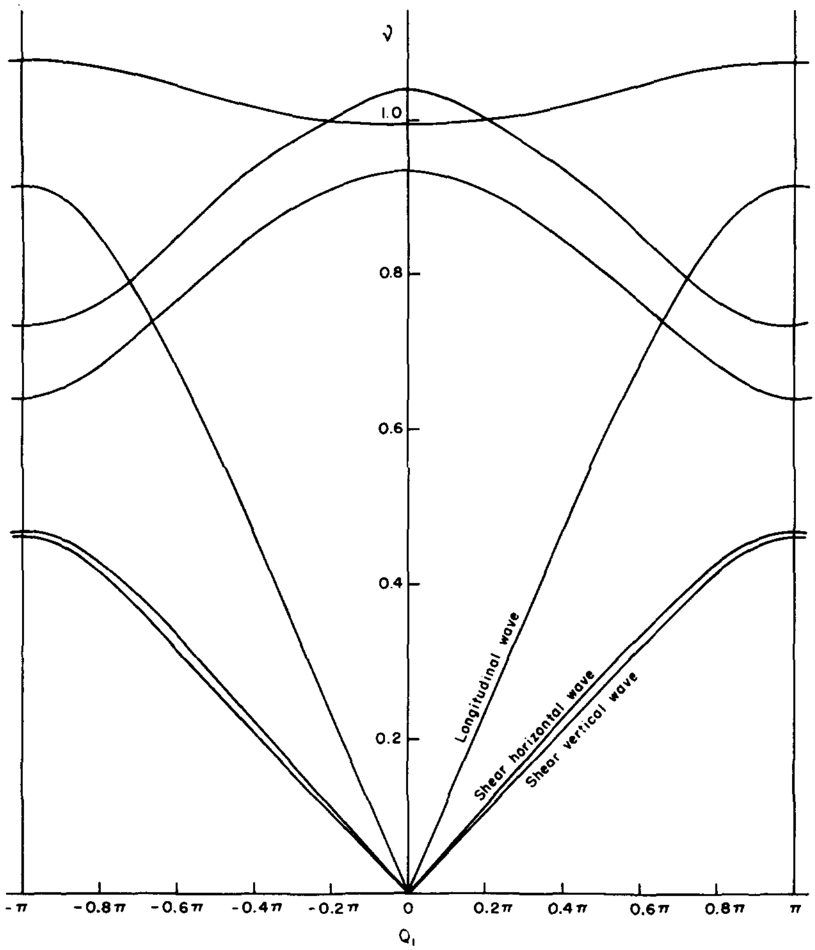

Fig. 7. Acoustical and optical brnaches for waves propagating in a composite with ellipsoidal inclusions ($q_2 = q_3 = 0$): $\theta = 3, \gamma = 50, \nu_M = 0.35, \nu_F = 0.3, n_0 = 1, m_0 = 1, n_2 = 0.9, m_2 = 1, \ell_2 = 0.6, M' = 1$.

which are obtained by rendering λ_N stationary; also, as has been proved in [5], $\bar{\sigma}_{jk}^{(p)}$ are orthogonal with respect to the weighting tensor D_{jkmn}. This procedure is illustrated in Table 1 for $\bar{M} = 9$. Note that the improved eigenfrequencies do *not* represent upper bounds.

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